Newton and the Binomial Theorem

Major Source: Journey Through Genius, William Dunham, chapter 7 (1990). A highly recommended book.

Suggestion: Don't read this without paper and pencil handy!

I would think (although I am constantly being surprised) that you have seen Pascal's Triangle before. I'll try to remind you here (try because my typing skills are always suspect):

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
…

Pascal's triangle has an amazing usefulness in a wide collection of applications. For our purposes, it contains the coefficients of the expansion of $(x+y)^n$. For example, $(x+y)^2 = x^2 + 2xy + y^2$ and $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

So, ho-hum, you have to find ALL the rows before the one you're interested in, so what's the point? You belong to such a spoiled generation! So –

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Extra Credit, part 1: Multiply out $(x+y)^4$ the long way (pencil and paper) and, after collecting terms, check your answer with the appropriate row of the triangle above. How did you do?

Accuracy hint => If x or y are actual numbers, write down the expansion first and only THEN substitute the number in for the variable. (Trust me – it makes sense as I'm trying to write it).

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But the ***cool part (+1)*** is that there is a closed form expression for any entry in the table! For example, the ***k***th entry in the ***n***th row is $\frac{n!}{k!(n-k)!} = \frac{n*(n-1)*(n-2)*...*1}{(k*(k-1)*...*1)((n-k)*(n-k-1)*...*1)}$. Umm … we need to call the top row of Pascal's triangle the "0th" row, but other than that it works well! So – if there is a question on the AMC exam (likely) or the SAT (less likely) that asks for the coefficient of the x^5y^7 term in the expansion of $(x+y)^{12}$ then you just do $\frac{12!}{5!*7!}$, cancel a whole bunch before you multiply (or use your calculator) and get the answer directly.

BTW – The ! notation is read "factorial" and just means 'keep multiplying by one less until you get to 1'. If you remember combinations, it's the same expression for "n choose k". I don't think I can type it with the tools I have. It looks like $(\frac{n}{k})$ but there is **NO** fraction line in the middle.

But the ***cool part (+2)*** is that Newton asked a simple question …. why does ***n*** have to be a positive integer? What happens if the '***n***' is **any** rational number?

And he decided that the formula is still true.

For sure, there is a minor problem – the terms (the coefficients actually) never stop! So, here is the general form of the binomial expansion:

$$(1+q)^r = 1 + rq + \frac{r(r-1)}{2}q^2 + \frac{r(r-1)(r-2)}{3*2}q^3 + \ldots$$

And Newton got an incredible amount of mileage out of this.

For example …. Note that $7 = 9(\frac{7}{9}) = 9(1-\frac{2}{9})$. So … $\sqrt{7} = \sqrt{9(1-\frac{2}{9})} = 3\sqrt{(1-\frac{2}{9})}$. Newton then used $q = -\frac{2}{9}$ and $r = \frac{1}{2}$ to get an approximation of $\sqrt{7}$. It converges extremely quickly – and was a better tool than the previous method (an educated 'guess and check' algorithm I would imagine).

Newton then used this (and the calculus!) to get an approximation of pi.

Extra Credit:

Use the binomial theorem to get an approximation for the $\sqrt{3}$. Check it with your calculator. How many terms do you need to consider for an accuracy of three decimal places?

Use the binomial theorem to get an infinite-term 'polynomial' for $\sqrt{(1-x)}$. You don't need to write out all the terms ☺ But make sure I know YOU know how to get each one!

AND write it up nicely for me ☺

Other sources:

http://mathworld.wolfram.com/PascalsTriangle.html

And let me know if you find other good ones.